COMPLEMENTED SUBSPACES OF LOCALLY CONVEX DIRECT SUMS OF BANACH SPACES

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Abstract. We show that a complemented subspace of a locally convex direct sum of an uncountable collection of Banach spaces is a locally convex direct sum of complemented subspaces of countable subsums. As a corollary we prove that a complemented subspace of a locally convex direct sum of arbitrary collection of $\ell_1(\Gamma)$-spaces is isomorphic to a locally convex direct sum of $\ell_1(\Gamma)$-spaces.

1. Introduction

In 1960 A. Pelczynski proved [4] that complemented subspaces of ℓ_1 are isomorphic to ℓ_1. In [3] G. Köthe generalized this result to the non-separable case. Later, while answering Köthe's question about precise description of projective spaces in the category of (LB)-spaces, P. Domański showed [2] that complemented subspaces of locally convex direct sums of countable collections of $\ell_1(\Gamma)$-spaces have the same structure, i.e. are isomorphic to locally convex direct sums of countable collections of $\ell_1(\Gamma)$-spaces.

Below we complete this series of statements by showing (Corollary 2.3) that countability assumption in Domański's result is not essential. More precisely, we prove that complemented subspaces of a locally convex direct sums of arbitrary collections of $\ell_1(\Gamma)$-spaces are isomorphic to locally convex direct sums of $\ell_1(\Gamma)$-spaces. This is obtained as a corollary of our main result (Theorem 2.2) stating that complemented subspaces of locally convex direct sums of arbitrary collections of Banach spaces are isomorphic to locally convex direct sums of complemented subspaces of countable subsums.

2. Results

Below we work with locally convex direct sums $\bigoplus\{B_t\colon t \in T\}$ of uncountable collections of Banach spaces B_t, $t \in T$. Recall that if $S \subseteq R \subseteq T$, then $\bigoplus\{B_t\colon t \in S\}$ can be canonically identified with the subspace

$$\Big\{\{x_t\colon t \in R\} \in \bigoplus\{B_t\colon t \in R\}\colon x_t = 0 \text{ for each } t \in R - S\Big\}$$

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of $\bigoplus\{B_t\colon t \in R\}$. The corresponding inclusion is denoted by i^R_S. The following statement is used in the proof of Theorem 2.2.

Proposition 2.1. *Let* $r\colon \bigoplus\{B_t\colon t \in T\} \to \bigoplus\{B_t\colon t \in T\}$ *be a continuous linear map of a locally convex direct sum of an uncountable collection of Banach spaces into itself. Let also* A *be a countable subset of* T*. Then there exists a countable subset* $S \subseteq T$ *such that* $A \subseteq S$ *and* $r\left(\bigoplus\{B_t\colon t \in S\}\right) \subseteq \bigoplus\{B_t\colon t \in S\}$.

Proof. Let $\exp_\omega T$ denote the set of all countable subsets of the indexing set T. Consider the following relation

$$\mathcal{L} = \left\{(A,C) \in (exp_\omega T)^2\colon A \subseteq C \text{ and } r\left(\bigoplus\{B_t\colon t \in A\}\right) \subseteq \bigoplus\{B_t\colon t \in C\}\right\}.$$

We need to verify the following three properties of the above defined relation.

Existence. If $A \in \exp_\omega T$, then there exists $C \in \exp_\omega T$ such that $(A,C) \in \mathcal{L}$.

Proof. First of all let us make the following observation.

Claim. *For each* $j \in T$ *there exists a finite subset* $C_j \subseteq T$ *such that* $r\left(B_j\right) \subseteq \bigoplus\{B_t\colon t \in C_j\}$.

Proof of Claim. The unit ball $K = \{x \in B_j\colon ||x||_j \leq 1\}$ (here $||\cdot||_j$ denotes the norm of the Banach space B_j) being bounded in B_j is, by [5, Theorem 6.3], bounded in $\bigoplus\{B_t\colon t \in T\}$. Continuity of r guarantees that $r(K)$ is also bounded in $\bigoplus\{B_t\colon t \in T\}$. Applying [5, Theorem 6.3] once again, we conclude that there exists a finite subset $C_j \subseteq T$ such that $r(K) \subseteq \bigoplus\{B_t\colon t \in C_j\}$. Finally the linearity of r implies that $r\left(B_j\right) \subseteq \bigoplus\{B_t\colon t \in C_j\}$ and proves the Claim.

Let now $A \in \exp_\omega T$. For each $j \in A$, according to Claim, there exists a finite subset $C_j \subseteq T$ such that $r\left(B_j\right) \subseteq \bigoplus\{B_t\colon t \in C_j\}$. Without loss of generality we may assume that $A \subseteq C_j$ for each $j \in A$. Let $C = \cup\{C_j\colon j \in A\}$. Clearly C is countable, $A \subseteq C$ and $r\left(B_j\right) \subseteq \bigoplus\{B_t\colon t \in C\}$ for each $j \in A$. This guarantees that $r\left(\bigoplus\{B_t\colon t \in A\}\right) \subseteq \bigoplus\{B_t\colon t \in C\}$ and shows that $(A,C) \in \mathcal{L}$.

Majorantness. If $(A,C) \in \mathcal{L}$, $D \in \exp_\omega T$ and $C \subseteq D$, then $(A,D) \in \mathcal{L}$.

Proof. Condition $(A,C) \in \mathcal{L}$ implies that $r\left(\bigoplus\{B_t\colon t \in A\}\right) \subseteq \bigoplus\{B_t\colon t \in C\}$. The inclusion $C \subseteq D$ implies that $\bigoplus\{B_t\colon t \in C\} \subseteq \bigoplus\{B_t\colon t \in D\}$. Consequently $r\left(\bigoplus\{B_t\colon t \in A\}\right) \subseteq \bigoplus\{B_t\colon t \in C\} \subseteq \bigoplus\{B_t\colon t \in D\}$, which means that $(A,D) \in \mathcal{L}$.

ω-closeness. Suppose that $(A_i, C) \in \mathcal{L}$ and $A_i \subseteq A_{i+1}$ for each $i \in \omega$. Then $(A,C) \in \mathcal{L}$, where $A = \cup\{A_i\colon i \in \omega\}$.

Proof. Consider the following inductive sequence

$$\bigoplus\{B_t\colon t\in A_0\}\xrightarrow{i_{A_0}^{A_1}}\cdots\to\bigoplus\{B_t\colon t\in A_i\}\xrightarrow{i_{A_i}^{A_{i+1}}}\bigoplus\{B_t\colon t\in A_{i+1}\}\to\cdots$$

limit of which is isomorphic to $\bigoplus\{B_t\colon t\in A\}$ (horizontal arrows represent canonical inclusions). Since $r\left(\bigoplus\{B_t\colon t\in A_i\}\right)\subseteq\bigoplus\{B_t\colon t\in C\}$ for each $i\in\omega$ (assumption $(A_i,C)\in\mathcal{L}$), it follows that

$$r\left(\bigoplus\{B_t\colon t\in A\}\right)=r\left(\operatorname{inj}\lim\left\{\bigoplus\{B_t\colon t\in A_i\},i_{A_i}^{A_{i+1}},\in\omega\right\}\right)\subseteq \bigoplus\{B_t\colon t\in C\}.$$

This obviously means that $(A,C)\in\mathcal{L}$ as required.

According to [1, Proposition 1.1.29] the set of $\mathcal{L}$-reflexive elements of $\exp_\omega T$ is cofinal in $\exp_\omega T$. An element $S\in\exp_\omega T$ is $\mathcal{L}$-reflexive if $(S,S)\in\mathcal{L}$. In our situation this means that the given countable subset A of T is contained in a larger countable subset S such that $r\left(\bigoplus\{B_t\colon t\in S\}\right)\subseteq\bigoplus\{B_t\colon t\in S\}$. Proof is completed. □

Theorem 2.2. *Let T be an uncountable set. A complemented subspace of a locally convex direct sum $\bigoplus\{B_t\colon t\in T\}$ of Banach spaces B_t, $t\in T$, is isomorphic to a locally convex direct sum $\bigoplus\{F_j\colon j\in J\}$, where F_j is a complemented subspace of the countable sum $\bigoplus\{B_t\colon t\in T_j\}$ where $|T_j|=\omega$ for each $j\in J$.*

Proof. Let X be a complemented subspace of the sum $B=\bigoplus\{B_t\colon t\in T\}$. Choose a continuous linear map $r\colon B\to X$ such that $r(x)=x$ for each $x\in X$. Let us agree that a subset $S\subseteq T$ is called r-admissible if $r\left(\bigoplus\{B_t\colon t\in S\}\right)\subseteq\bigoplus\{B_t\colon t\in S\}$.

For a subset $S\subseteq T$, let $X_S=r\left(\bigoplus\{B_t\colon t\in S\}\right)$.

Claim 1. *If $S\subseteq T$ is an r-admissible, then $X_S=X\bigcap\left(\bigoplus\{B_t\colon t\in S\}\right)$.*

Proof. Indeed, if $y\in X_S$, then there exists a point $x\in\bigoplus\{B_t\colon t\in S\}$ such that $r(x)=y$. Since S is r-admissible, it follows that

$$y=r(x)\in r\left(\bigoplus\{B_t\colon t\in S\}\right)\subseteq\bigoplus\{B_t\colon t\in S\}.$$

Clearly, $y\in X$. This shows that $X_S\subseteq X\bigcap\left(\bigoplus\{B_t\colon t\in S\}\right)$.

Conversely, if $y\in X\bigcap\left(\bigoplus\{B_t\colon t\in S\}\right)$, then $y\in X$ and hence, by the property of r, $y=r(y)$. Since $y\in\bigoplus\{B_t\colon t\in S\}$, it follows that $y=r(y)\in r\left(\bigoplus\{B_t\colon t\in S\}\right)=X_S$.

Claim 2. *The union of an arbitrary collection of r-admissible subsets of T is r-admissible.*

Proof. Straightforward verification based of the definition of the r-admissibility.

Claim 3. *Every countable subset of T is contained in a countable r-admissible subset of T.*

Proof. This follows from Proposition 2.1 applied to the map r.

Claim 4. *If $S \subseteq T$ is an r-admissible subset of T, then $r_S(x) = x$ for each point $x \in X_S$, where $r_S = r \Big| \left(\bigoplus\{B_t \colon t \in S\}\right) : \bigoplus\{B_t \colon t \in S\} \to X_S$.*

Proof. This follows from the corresponding property of the map r.

Before we state the next property of r-admissible sets note that if $S \subseteq R \subseteq T$, then the map

$$\pi^R_S \colon \bigoplus\{B_t \colon t \in R\} \to \bigoplus\{B_t \colon t \in S\},$$

defined by letting

$$\pi^R_S\left(\{x_t \colon t \in R\}\right) = \begin{cases} x_t, & \text{if } t \in S \\ 0\,, & \text{if } t \in R - S, \end{cases}$$

is continuous and linear.

Claim 5. *Let S and R are r-admissible subsets of T and $S \subseteq R$. Then X_S is a complemented subspace in X_R and $X_R/_{X_S}$ is a complemented subspace in $\bigoplus\{B_t \colon t \in R - S\}$.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccc}
\bigoplus\{B_t \colon t \in R\} & \xrightarrow{r_R} & X_R \\
\Big\downarrow{\scriptstyle \pi^R_{R-S}} & & \Big\downarrow{\scriptstyle p} \\
\oplus\{B_t \colon t \in R - S\} = \oplus\{B_t \colon t \in R\}/_{\oplus\{B_t \colon t \in S\}} & \xrightarrow{q} & X_R/_{X_S}
\end{array}$$

in which p is the canonical map and q is defined on cosets by letting (recall that $r_R\left(\bigoplus\{B_t \colon t \in S\}\right) = X_S$)

$$q\left(x + \bigoplus\{B_t \colon t \in S\}\right) = r_R(x) + X_S \ \text{ for each } \ x \in \bigoplus\{B_t \colon t \in R\}.$$

Let us denote by $i_R \colon X_R \hookrightarrow \bigoplus\{B_t \colon t \in R\}$ the natural inclusion and consider a map

$$j \colon X_R/_{X_S} \to \oplus\{B_t \colon t \in R\}/_{\oplus\{B_t \colon t \in S\}}$$

defined by letting (in terms of cosets)

$$j\,(x + X_S) = i(x) + \bigoplus\{B_t\colon t \in S\} \text{ for each } x \in X_R.$$

Note that $q \circ j = \mathrm{id}_{X_R/X_S}$ (this follows from the equality $r_R \circ i = \mathrm{id}_{X_R}$). In particular, this shows that X_R/X_S is isomorphic to a complemented subspace of $\bigoplus\{B_t\colon t \in R - S\}$.

Finally consider the composition $r_R \circ i^R_{R-S} \circ j\colon X_R/X_S \to X_R$ and note that

$$p \circ (r_R \circ i^R_{R-S} \circ j) = p \circ r_R \circ i^R_{R-S} \circ j = q \circ \pi^R_{R-S} \circ i^R_{R-S} \circ j = q \circ \mathrm{id} \circ j = \mathrm{id}_{X_R/X_S}.$$

This shows that X_S is a complemented subspace of X_R and completes the proof of Claim 5.

Let $|T| = \tau$. Then we can write $T = \{t_\alpha\colon \alpha < \tau\}$. Since the collection of countable r-admissible subsets of T is cofinal in $\exp_\omega T$ (see Claim 3), each element $t_\alpha \in T$ is contained in a countable r-admissible subset $A_\alpha \subseteq T$. According to Claim 2, the set $T_\alpha = \bigcup\{A_\beta\colon \beta \leq \alpha\}$ is r-admissible for each $\alpha < \tau$. Consider the inductive system $\mathcal{S} = \{X_\alpha, i^{\alpha+1}_\alpha, \tau\}$, where $X_\alpha = X_{T_\alpha} = X \bigcap r\,(\bigoplus\{B_t\colon t \in T_\alpha\})$ (see Claim 1) and $i^{\alpha+1}_\alpha\colon X_\alpha \to X_{\alpha+1}$ denotes the natural inclusion for each $\alpha < \tau$. For a limit ordinal number $\beta < \tau$ the space X_β is isomorphic to the limit space of the direct system $\{X_\alpha, i^{\alpha+1}_\alpha, \alpha < \beta\}$ (verification of this fact is based on Claim 4 coupled with the fact that $\bigoplus\{B_t\colon t \in T_\beta\}$ is isomorphic to the limit of the direct system $\left\{\bigoplus\{B_t\colon t \in T_\alpha\}, i^{T_{\alpha+1}}_{T_\alpha}, \alpha < \beta\right\}$). In particular, X is isomorphic to the limit of the inductive system $\{X_\alpha, i^{\alpha+1}_\alpha, \alpha < \tau\}$.

For each $\alpha < \tau$, according to Claim 5, the inclusion $i^{\alpha+1}_\alpha\colon X_\alpha \to X_{\alpha+1}$ is isomorphic to the inclusion $X_\alpha \hookrightarrow X_\alpha \bigoplus X_{\alpha+1}/X_\alpha$. In this situation the straightforward transfinite induction shows that X is isomorphic to the locally convex direct sum $X_0 \bigoplus (\bigoplus\{X_{\alpha+1}/X_\alpha\colon \alpha < \tau\})$.

By construction, the set T_0 is countable and X_0 is a complemented subspace of $\bigoplus\{B_t\colon t \in T_0\}$. Note also that for each $\alpha < \tau$ the set $T_{\alpha+1} - T_\alpha = A_{\alpha+1}$ is countable and $X_{\alpha+1}/X_\alpha$ is a complemented subspace of $\bigoplus\{B_t\colon t \in A_{\alpha+1}\}$. This completes the proof of Theorem 2.2. □

The following statement, as was noted in the Introduction, provides a complete description of complemented subspaces of locally convex direct sums of uncountable collections of $\ell_1(\Gamma)$-spaces.

Corollary 2.3. *Let X be a complemented subspace of $\bigoplus\{\ell_1(\Gamma_t)\colon t \in T\}$. Then X is isomorphic to $\bigoplus\{\ell_1(\Lambda_i)\colon i \in I\}$.*

Proof. For countable T results follows from [3] and [2]. Let now T is uncountable and X be a complemented subspace of a locally convex direct sum $\bigoplus\{\ell_1(\Gamma_t)\colon t \in T\}$. By Theorem 2.2, X is isomorphic to a locally convex direct sum $\bigoplus\{F_j\colon j \in J\}$, where F_j is a complemented subspace of the countable sum $\bigoplus\{\ell_1(\Gamma_t)\colon t \in T_j\}$ where $|T_j| = \omega$ for each $j \in J$. According to [2], $F_j = \bigoplus\{\ell_1(\Lambda_t)\colon\ \in t \in T_j\}$

for each $j \in J$. Consequently, X is isomorphic to the locally convex direct sum $\bigoplus\{\bigoplus\{\ell_1(\Lambda_t \colon t \in T_j\} \colon j \in J\} = \bigoplus\{\ell_1(\Lambda_t) \colon t \in \cup\{T_j \colon j \in J\}\}$ as required. □

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